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ON WAVE PROPAGATION IN INHOMOGENEOUS ELASTIC MEDIA

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Abstract—Wave propagation in an inhomogeneous elastic rod or slab is considered. The governing equations are written in a matrix form and transformations are sought which reduce the system to a form associated with the wave equation. Integration of the system is then immediate. It is shown that such reduction may be achieved subject to a function involving the density and elastic parameters of the material adopting certain multi-parameter forms. These parameters are available for fitting to the behaviour of a variety of inhomogeneous elastic materials. A specific initial boundary value problem is solved by utilising the present method.

1. INTRODUCTION

One-dimensional wave propagation in an inhomogeneous elastic material has been investigated by Payton [1], and Eason [2]. In the latter paper, the elastic parameters were assumed to depend on one spatial co-ordinate alone; the stress and displacement components were taken to be dependent on this space co-ordinate and time. It was shown that both for longitudinal wave propagation in a rod and also for the propagation of shear waves under conditions of cylindrical or spherical symmetry, the basic equations reduce to the wave equation with variable wave speed. The latter equation has been discussed by various authors in connection with other physical situations (see [3–5]). Here, a novel approach is adopted in which matrix transformations are generated which reduce the equation to the conventional wave equation. Consequences of this reduction are then developed.

2. THE GOVERNING EQUATIONS

Wave propagation along an inhomogeneous elastic rod or in an inhomogeneous elastic slab is considered. The relevant stress–strain relation is

$$\sigma = \xi \frac{\partial u}{\partial x}, \tag{2.1}$$

where x is the space co-ordinate measured along the rod, σ is the stress and u is the displacement; here $\xi \equiv E$ in the case of the rod and $\xi \equiv \lambda + 2\mu$ for wave propagation in a slab (E is Young's modulus, λ and μ are the Lamé constants). In the absence of body forces, the equation of motion becomes

$$\frac{\partial \sigma}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2}, \tag{2.2}$$

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where ρ is the density of the medium, and t denotes time. It is here assumed that both ρ and E are independent of t. Combination of (2.1) and (2.2) yields

$$\frac{\partial}{\partial x}\left(\xi\frac{\partial u}{\partial x}\right) = \rho\frac{\partial^2 u}{\partial t^2}. \tag{2.3}$$

If the independent variable x is now changed according to the transformation

$$y = \int_{x_0}^{x} (\rho/\xi)^{1/2}\,\mathrm{d}x \tag{2.4}$$

(2.3) becomes

$$\frac{\partial}{\partial y}\left[(\rho\xi)^{1/2}\frac{\partial u}{\partial y}\right] = (\rho\xi)^{1/2}\frac{\partial^2 u}{\partial t^2}. \tag{2.5}$$

In (2.4), x_0 is a convariant reference co-ordinate. Introduction of ε and v defined by

$$\varepsilon = (\rho\xi)^{1/2}\frac{\partial u}{\partial y},\ v = \frac{\partial u}{\partial t}, \tag{2.6}$$

now provides a convenient matrix system descriptive of the wave propagation, namely

$$\Omega_y = H\Omega_t, \tag{2.7}$$

where subscripts denote partial derivatives and the matrices are defined by

$$\Omega = \begin{pmatrix}\varepsilon\\ v\end{pmatrix}, \qquad H = \begin{pmatrix}0 & K^{1/2}\\ K^{-1/2} & 0\end{pmatrix}, \qquad K = \rho E. \tag{2.8}$$

3. THE MATRIX TRANSFORMATIONS

Matrix transformations of the form

$$\Omega'_{y'} = A\Omega_y + B\Omega, \qquad |A| \neq 0, \tag{3.1}$$

$$\Omega'_{y'} = \tilde{A}\Omega_t + \tilde{B}\Omega_t \qquad |\tilde{A}| \neq 0, \tag{3.2}$$

$$y' = y, \qquad t' = t \tag{3.3}$$

are now introduced where A, B, $\tilde{A}$, $\tilde{B}$ are, in turn, 2×2 matrices $[a^i_j]$, $[b^i_j]$, $[\tilde{a}^i_j]$, $[\tilde{b}^i_j]$ $i, j = 1, 2$ with entries functions of y. Transformations of this type are sought which transform

$$\Omega_y = H\Omega_t \rightarrow \Omega'_{y'} = H'\Omega'_{t'}, \tag{3.4}$$

where H is defined by $(2.8)_2$ and H' adopts a form associated with the wave equation, namely

$$H' = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{3.5}$$

Transformations of the form (3.1–3.3) were first introduced by Loewner [6] in connection with the reduction to canonical form of the hodograph equations in subsonic, transonic and supersonic flow It emerges that reduction may be achieved subject to the real gas pressure-density relation being approximated by certain multi-parameter forms. Various important approximations of gasdynamics such as the well-known Kármán-Tsien relation may be extracted as particular cases of the theory. In analogous fashion, in this paper, it is shown that reduction of the system (2.7) to one associated with the wave equation may be made provided $\rho\xi$ can be approximated by certain forms.

It is assumed that ε, v, ε', v' have continuous mixed second order derivatives with respect to the independent variables y, t so that the commutativity conditions

$$\Omega_{yt} = \Omega_{ty}, \qquad \Omega'_{y't'} = \Omega'_{t'y'}. \tag{3.6), (3.7}$$

obtain. Thus, employing these conditions and the relations (3.1–3.3) it is seen that

$$(A - \tilde{A})\Omega_{yt} - \tilde{B}\Omega_y + (B - \tilde{A}_y)\Omega_t - \tilde{B}_y\Omega = 0. \tag{3.8}$$

Since $\Omega_y = H\Omega_t$, (3.8) is identically satisfied by setting

$$A = \tilde{A}, \tag{3.9}$$

$$-\tilde{B}H + (B - \tilde{A}_y) = 0, \tag{3.10}$$

$$\tilde{B}_y = 0. \tag{3.11}$$

Returning to (3.1) and (3.2), it follows that, if A is non-singular

$$\Omega'_{y'} - H'\Omega'_{t'} = A[\Omega_y - A^{-1}H'A\Omega_t] + (B - H'\tilde{B})\Omega, \tag{3.12}$$

whence, setting

$$A^{-1}H'A = H, \qquad B = H'\tilde{B}, \tag{3.13), (3.14}$$

the system $\Omega_y = H\Omega_t$ is transformed to the associated system $\Omega'_{y'} = H'\Omega'_{t'}$ and conversely via the transformations (3.1–3.3) subject to the conditions (3.9–3.14) prevailing. Thus the transformation (3.4) is achieved via the relations

$$\Omega'_y = \tilde{A}\Omega_y + H'\tilde{B}\Omega, \tag{3.15}$$

$$\Omega'_t = \tilde{A}\Omega_y + \tilde{B}\Omega, \tag{3.16}$$

where $\tilde{B}$ is a constant matrix and

$$\tilde{A}_y - H'\tilde{B} + \tilde{B}\tilde{A}^{-1}H'\tilde{A} = 0. \tag{3.17}$$

It is now necessary to specialise the matrix $A = \tilde{A}$ so that the property of zero diagonal elements is preserved under the mapping $H \to H'$. From (3.13) it is clear that this property is invariant if (but not only if) $A = \tilde{A}$ adopts the diagonal form

$$A = \begin{pmatrix} a_1^1 & 0 \\ 0 & a_2^2 \end{pmatrix}, \tag{3.18}$$

in which case

$$H' = AHA^{-1} = \begin{pmatrix} 0 & a_1^1 h_2^1 a_2^2 \\ a_2^2 h_1^2 a_1^1 & 0 \end{pmatrix}, \tag{3.19}$$

while (3.17) yields

$$(a_1^1)_y - h_2^{1\prime}\tilde{b}_1^2 + h_1^{2\prime}\tilde{b}_2^1(a_1^1/a_2^2) = 0, \tag{3.20}$$

$$(a_2^2)_y - h_1^{2\prime}\tilde{b}_2^1 + h_2^{1\prime}\tilde{b}_1^2(a_2^2/a_1^1) = 0. \tag{3.21}$$

Combination of equations (3.20) and (3.21) shows that

$$\det A = a_1^1 a_2^2 = \text{constant} = \lambda,\ \lambda \neq 0, \tag{3.22}$$

whence, the system may be reduced to a single Riccati equation in a_1^1 or a_2^2. In particular, the Riccati equation in a_1^1 is, on setting $h_2^{1\prime} = h_1^{2\prime} = 1$,

$$(a_1^1)_y + \alpha(a_1^1)^2 + \beta = 0, \qquad (\alpha = b_2^2\lambda^{-1}, \qquad \beta = -b_1^1) \tag{3.23}$$

Thus,

(a) If $\beta = 0$,

$$a_1^1 = 1/(\alpha y + \varepsilon),$$

(b) If $\alpha = 0$,

$$a_1^1 = -\beta y + \delta,$$

(c) If $\beta/\alpha > 0$,

$$a_1^1 = (\beta/\alpha)^{1/2} \cot\{(\beta/\alpha)^{1/2}(\alpha y + \zeta)\}.$$

(d) If $\beta/\alpha < 0$

$$a_1^1 = (-\beta/\alpha)^{1/2} \tanh\{(-\beta/\alpha)^{1/2}(\alpha y + \eta)\}$$

where δ, ε, ζ, η are arbitrary constants of integration.

Now, since

$$K = (a_2^2/a_1^1)^2 = \lambda^2/(a_1^1)^4,$$

it is seen that reduction of the system (2.8) to one associated with the wave equation may be made provided $K = \rho\xi$ adopts one of the forms

(a) $\lambda^2[\alpha y + \varepsilon]^4$ (b) $\lambda^2/[-\beta y + \delta]^4$,

(c) $(\lambda^2\alpha^2/\beta^2)\tan^4\{(\beta/\alpha)^{1/2}(\alpha y + \zeta)\}$ (d) $(\lambda^2\alpha^2/\beta^2)\coth^4\{(-\beta/\alpha)^{1/2}(\alpha y + \eta)\}$.

4. INTEGRATION OF THE BASIC EQUATIONS

The matrix transformations (3.16) now yield

$$\left.\begin{aligned} \begin{pmatrix} \varepsilon' \\ v' \end{pmatrix}_{y'} &= \begin{pmatrix} a_1^1 & 0 \\ 0 & a_2^2 \end{pmatrix} \begin{pmatrix} \varepsilon \\ v \end{pmatrix}_y + \begin{pmatrix} b_1^1 & 0 \\ 0 & b_2^2 \end{pmatrix} \begin{pmatrix} \varepsilon \\ v \end{pmatrix}, \\ \begin{pmatrix} \varepsilon' \\ v' \end{pmatrix}_{t'} &= \begin{pmatrix} a_1^1 & 0 \\ 0 & a_2^2 \end{pmatrix} \begin{pmatrix} \varepsilon \\ v \end{pmatrix}_t + \begin{pmatrix} 0 & b_2^2 \\ b_1^1 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon \\ v \end{pmatrix}, \\ y' &= y,\ t' = t \end{aligned}\right\} \tag{4.1}$$

where a_1^1 adopts one of the forms (a–d) of the preceding section, $a_2^2 = \lambda/a_1^1$ and b_1^1, b_2^2 are constants. The cases (a–d) are now investigated in turn.

(a)

The system (4.1) gives

$$\varepsilon'_y = [\lambda/(b_2^2 y + \lambda\varepsilon)]\varepsilon_y, \qquad \varepsilon'_t = [\lambda/(b_2^2 y + \lambda\varepsilon)]\varepsilon_t + b_2^2 v, \tag{4.2}$$

$$v'_y = [b_2^2 y + \lambda\varepsilon]v_y + b_2^2, \qquad v'_t = (b_2^2 y + \lambda\varepsilon)v_t. \tag{4.3}$$

The relations (4.3) provide

$$v' = (b_2^2 y + \lambda\varepsilon)v,$$

so that

$$v = (b_2^2 y + \lambda\varepsilon)^{-1}[H(\bar{\xi}) + J(\bar{\eta})], \tag{4.4}$$

where H, J are arbitrary functions of $\bar{\xi}$, $\bar{\eta}$ respectively with

$$\bar{\xi} = (y + t)/2, \qquad \bar{\eta} = (y - t)/2. \tag{4.5), (4.6}$$

Moreover, equations (4.2) show that

$$\varepsilon'_y = \tfrac{1}{2}[H'(\bar{\xi}) - J'(\bar{\eta})], \qquad \varepsilon'_t = \tfrac{1}{2}[H'(\bar{\xi}) + J'(\bar{\eta})],$$

whence

$$\varepsilon' = H(\bar{\xi}) - J(\bar{\eta}),$$

so that, employing (4.2),

$$\varepsilon_y = \tfrac{1}{2}(b_2^2 y + \lambda\varepsilon)\lambda^{-1}[H'(\bar{\xi}) - J'(\bar{\eta})],$$
$$\varepsilon_t = \tfrac{1}{2}(b_2^2 y + \lambda\varepsilon)\lambda^{-1}[H'(\bar{\xi}) + J'(\bar{\eta})] - b_2^2\lambda^{-1}[H(\bar{\xi}) + J(\bar{\eta})].$$

Integration of the latter pair of relations gives

$$\varepsilon = b_2^2\lambda^{-1}\Big[\{\bar{\xi} + \bar{\eta} + \lambda\varepsilon/b_2^2\}[H(\bar{\xi}) - J(\bar{\eta})] - 2\Big\{\int H(\bar{\xi})\,\mathrm{d}\bar{\xi} - \int J(\bar{\eta})\,\mathrm{d}\bar{\eta}\Big\}\Big]. \tag{4.7}$$

Thus, when the product $\rho\xi$ takes the form

$$\rho\xi = \lambda^2[\alpha y + \varepsilon]^4, \qquad (\alpha = b_2^2\lambda^{-1}),$$

the system (2.8) may be integrated to provide the expressions (4.7) and (4.4) for ε and v respectively.

(b)

In this case, from (4.1),

$$\varepsilon'_y = (b_1^1 y + \delta)\varepsilon_y + b_1^1\varepsilon, \qquad \varepsilon'_t = (b_1^1 y + \delta)\varepsilon_t, \tag{4.8}$$

$$v'_y = [\lambda/(b_1^1 y + \delta)]v_y, \qquad v'_t = [\lambda/(b_1^1 y + \delta)]v_t + b_1^1\varepsilon. \tag{4.9}$$

Integration of the relations (4.8) yields

$$\varepsilon' = (b_1^1 y + \delta)\varepsilon,$$

whence

$$\varepsilon = (b_1^1 y + \delta)^{-1}[F(\bar{\xi}) + G(\bar{\eta})], \tag{4.10}$$

where F, G are arbitrary functions of their indicated arguments. Further, from (4.9),

$$v'_y = [\lambda/(b_1^1 y + \delta)]v_y = [b_1^1 y + \delta]\varepsilon_t = \tfrac{1}{2}[F'(\bar{\xi}) - G'(\bar{\eta})],$$
$$v'_t = [\lambda/(b_1^1 y + \delta)]v_t + b_1^1\varepsilon = [b_1^1 y + \delta]\varepsilon_y + b_1^1\varepsilon = \tfrac{1}{2}[F'(\bar{\xi}) + G'(\bar{\eta})],$$

so that

$$v' = F(\bar{\xi}) - G(\bar{\eta}),$$

and

$$v_y = \tfrac{1}{2}(b_1^1 y + \delta)\lambda^{-1}[F'(\xi) - G'(\bar{\eta})], \tag{4.11}$$

$$v_t = \tfrac{1}{2}(b_1^1 y + \delta)\lambda^{-1}[F'(\xi) + G'(\bar{\eta})] - b_1^1\lambda^{-1}[F(\xi) + G(\bar{\eta})]. \tag{4.12}$$

In terms of the variables $\bar{\xi}$, $\bar{\eta}$, equations (4.11) and (4.12) may be written in the form

$$v_{\bar{\xi}} = [b_1^1(\bar{\xi} + \bar{\eta}) + \delta]\lambda^{-1}F'(\xi) - b_1^1\lambda^{-1}[F(\xi) + G(\bar{\eta})],$$

$$v_{\bar{\eta}} = -[b_1^1(\xi + \bar{\eta}) + \delta]\lambda^{-1}G'(\bar{\eta}) + b_1^1\lambda^{-1}[F(\xi) + G(\bar{\eta})],$$

whence, on integration,

$$v = b_1^1\lambda^{-1}\Big[\{\xi + \bar{\eta} + \delta/b_1^1\}[F(\xi) - G(\bar{\eta})] - 2\Big\{\int F(\xi)\, d\bar{\xi} - \int G(\bar{\eta})\, d\bar{\eta}\Big\}\Big]. \tag{4.13}$$

Consequently, it is seen that, when $\rho\xi$ adopts the form

$$\rho\xi = \lambda^2/[-\beta y + \delta]^2, \qquad (\beta = -b_1^1),$$

the system (2.8) integrates the expressions for ε and v being given by (4.10) and (4.13) respectively.

(c)

If $L(\xi)$ and $M(\bar{\eta})$ are arbitrary functions of their respective arguments,

$$\varepsilon' = L(\xi) + M(\bar{\eta}), \qquad v' = L(\xi) - M(\bar{\eta})$$

and the relations (4.1) provide, in this case

$$\varepsilon_y' = \tfrac{1}{2}[L'(\xi) + M'(\bar{\eta})] = a_1^1\varepsilon_y + b_1^1\varepsilon = a_2^2 v_t + b_1^1\varepsilon,$$

$$\varepsilon_t' = \tfrac{1}{2}[L'(\xi) - M'(\bar{\eta})] = a_2^2 v_y + b_2^2 v = a_1^1\varepsilon_t + b_2^2 v,$$

whence,

$$\frac{\partial}{\partial y}[\varepsilon\cos\{(\beta/\alpha)^{1/2}(\alpha y + \xi)\}] = \tfrac{1}{2}(\alpha/\beta)^{1/2}[L'(\xi) + M'(\bar{\eta})]\sin\{(\beta/\alpha)^{1/2}(\alpha y + \xi)\}, \tag{4.14}$$

and

$$\frac{\partial}{\partial y}[v\sin\{(\beta/\alpha)^{1/2}(\alpha y + \xi)\}] = \tfrac{1}{2}(\beta/\alpha)^{1/2}\lambda^{-1}[L'(\xi) - M'(\bar{\eta})]\cos\{(\beta/\alpha)^{1/2}(\alpha y + \xi)\}. \tag{4.15}$$

On integration, (4.14) and (4.15) yield

$$\begin{aligned}\varepsilon = (\alpha/\beta)^{1/2}[(L(\xi) + M(\bar{\eta}))\tan\{(\beta/\alpha)^{1/2}(\alpha y + \zeta)\} - \\ - (\alpha\beta)^{1/2}\sec\{(\beta/\alpha)^{1/2}(\alpha y + \zeta)\}\int(L(\xi) + M(\bar{\eta}))\cos\{(\beta/\alpha)^{1/2}(\alpha y + \zeta)\}\, dy] + \\ + T_1(t)\sec\{(\beta/\alpha)^{1/2}(\alpha y + \zeta)\},\end{aligned} \tag{4.16}$$

$$\begin{aligned}v = (\beta/\alpha)^{1/2}\lambda^{-1}[(L(\xi) - M(\bar{\eta}))\cot\{(\beta/\alpha)^{1/2}(\alpha y + \zeta)\} + \\ + (\alpha\beta)^{1/2}\operatorname{cosec}\{(\beta/\alpha)^{1/2}(\alpha y + \zeta)\}\int(L(\xi) - M(\bar{\eta}))\sin\{(\beta/\alpha)^{1/2}(\alpha y + \zeta)\}\, dy] \\ + T_2(t)\operatorname{cosec}\{(\beta/\alpha)^{1/2}(\alpha y + \zeta)\},\end{aligned} \tag{4.17}$$

where T_1, T_2 are functions of t. Moreover, the relations

$$\varepsilon_y = [\lambda/(a_1^1)^2]v_t, \qquad v_y = [(a_1^1)^2/\lambda]\varepsilon_t,$$

show that

$$T_1 = \lambda\alpha^{1/2}\beta^{-3/2}T_2', \qquad T_1' = -\lambda\alpha^{3/2}\beta^{-1/2}T_2,$$

whence, since $\alpha\beta > 0$ in this case,

$$T_1(t) = A\cos[(\alpha\beta)^{1/2}t] + B\sin[(\alpha\beta)^{1/2}t], \tag{4.18}$$

$$T_2(t) = \lambda^{-1}(\beta/\alpha)\{A\sin[(\alpha\beta)^{1/2}t] - B\cos[(\alpha\beta)^{1/2}t]\}, \tag{4.19}$$

where A, B are arbitrary constants of integration.

(d)

Setting

$$\varepsilon' = P(\xi) + Q(\bar{\eta}), \qquad v' = P(\xi) - Q(\bar{\eta}),$$

by virtue of the relations (4.1),

$$\frac{\partial}{\partial y}[\varepsilon\sinh\{(-\beta/\alpha)^{1/2}(\alpha y + \gamma)\}] = \tfrac{1}{2}(-\alpha/\beta)^{1/2}[P'(\xi) + Q'(\bar{\eta})]\cosh\{(-\beta/\alpha)^{1/2}(\alpha y + \gamma)\},$$

$$\frac{\partial}{\partial y}[v\cosh\{(-\beta/\alpha)^{1/2}(\alpha y + \gamma)\}] = \tfrac{1}{2}\lambda^{-1}(-\alpha/\beta)^{1/2}[P'(\xi) - Q'(\bar{\eta})]\sinh\{(-\beta/\alpha)^{1/2}(\alpha y + \gamma)\}.$$

On integration, it is seen that

$$\begin{aligned}\varepsilon = (-\alpha/\beta)^{1/2}[(P(\xi) + Q(\bar{\eta}))\coth\{(-\beta/\alpha)^{1/2}(\alpha y + \gamma)\} - \\ - (-\alpha\beta)^{1/2}\operatorname{cosech}\{(-\beta/\alpha)^{1/2}(\alpha y + \gamma)\}\int(P(\xi) + Q(\bar{\eta}))\sinh\{(-\beta/\alpha)^{1/2}(\alpha y + \gamma)\}\,dy] + \\ + T_3(t)\operatorname{cosech}\{(-\beta/\alpha)^{1/2}(\alpha y + \gamma)\},\end{aligned} \tag{4.20}$$

$$\begin{aligned}v = (-\alpha/\beta)^{1/2}\lambda^{-1}[(P(\xi) - Q(\bar{\eta}))\tanh\{(-\beta/\alpha)^{1/2}(\alpha y + \gamma)\} - \\ - (-\alpha\beta)^{1/2}\operatorname{sech}\{(-\beta/\alpha)^{1/2}(\alpha y + \gamma)\}\int(P(\xi) - Q(\bar{\eta}))\cosh\{(-\beta/\alpha)^{1/2}(\alpha y + \gamma)\}\,dy] + \\ + T_4(t)\operatorname{sech}\{(-\beta/\alpha)^{1/2}(\alpha y + \gamma)\}.\end{aligned} \tag{4.21}$$

The relations

$$\varepsilon_y = [\lambda/(a_1^1)^2]v_t, \qquad v_y = [(a_1^1)^2/\lambda]\varepsilon_t,$$

further show that

$$T_3 = C\cosh[(-\alpha\beta)^{1/2}t] + D\sinh[(-\alpha\beta)^{1/2}t], \tag{4.22}$$

$$T_4 = -\lambda^{-1}(\beta/\alpha)\{C\sinh[(-\alpha\beta)^{1/2}t] + D\cosh[(-\alpha\beta)^{1/2}t]\}, \tag{4.23}$$

where C, D are constants of integration.

5. WAVE PROPAGATION IN A SEMI-INFINITE ROD

The application of the preceding work to a specific problem involving wave propagation in an inhomogeneous semi-infinite rod is now considered. The semi-infinite rod $x > x_0$ is subjected to the displacement

$$u = H(t) \quad \text{at} \quad x = x_0, \tag{5.1}$$

where $H(t)$ is the step function. We suppose the density ρ and Young's modulus $\xi = E$ adopt the respective forms

$$E = E_0(1 + x)^m, \tag{5.2}$$

$$\rho = \rho_0(1 + x)^m \tag{5.3}$$

where $m = \pm 2$ and E_0 and ρ_0 are constants. Then, using (2.4),

$$y = (\rho_0/E_0)^{1/2}(x - x_0). \tag{5.4}$$

We consider the case $m = 2$ and $m = -2$ separately.

When $m = 2$ it is seen that ρE adopts the form associated with case (*a*), where

$$\alpha = (E_0/\rho_0)^{1/2}, \qquad \varepsilon = 1 + x_0, \qquad \lambda = (E_0\,\rho_0)^{1/2}. \tag{5.5}$$

Now in this case ε and v are given by (4.7) and (4.4) respectively and these equations may be readily integrated to give the displacement in the form

$$u = \lambda^{-1}(\alpha y + \varepsilon)^{-1}[V(\bar{\xi}) + W(\bar{\eta})], \tag{5.6}$$

where

$$V'(\bar{\xi}) = 2H(\bar{\xi}), \qquad W'(\bar{\eta}) = -2J(\bar{\eta}). \tag{5.7}$$

Recalling that $\bar{\xi} = (y + t)/2$ and $\bar{\eta} = (y - t)/2$, it follows that an appropriate choice of $V(\bar{\xi})$ and $W(\bar{\eta})$ is

$$V(\bar{\xi}) = 0, \qquad W(\bar{\eta}) = \lambda\varepsilon H(-2\bar{\eta}). \tag{5.8}$$

Thus, substitution into (5.6) shows that, on employing (5.4) and (5.5),

$$u = \frac{1 + x_0}{1 + x} H(\tau), \tag{5.9}$$

where

$$\tau = -2\bar{\eta} = t - \left(\frac{\rho_0}{E_0}\right)^{1/2}(x - x_0). \tag{5.10}$$

Thus, when the material parameters E and ρ increase with x according to (5.2) and (5.3) (with $m = 2$), it is apparent that the pulse propagates with constant speed. Also, the amplitude of the displacement given by (5.9) tends to zero as $x \to \infty$.

When $m = -2\rho E$ adopts the form associated with case (*b*), where

$$\beta = -(E_0/\rho_0)^{1/2}, \qquad \delta = 1 + x_0, \qquad \lambda = (E_0\,\rho_0)^{1/2}. \tag{5.11}$$

Integrating (4.10) and (4.13) we obtain an expression for the displacement which can be written in the form

$$u = \lambda^{-1}\left\{[b_1^1(\bar{\xi}+\bar{\eta})+\delta](M(\bar{\xi})+N(\bar{\eta})) - 2b_1^1\int M(\bar{\xi})\,d\bar{\xi} - 2b_1^1\int N(\bar{\eta})\,d\bar{\eta}\right\}, \qquad (5.12)$$

where

$$M'(\bar{\xi}) = 2F(\bar{\xi}), \qquad N'(\bar{\eta}) = 2G(\bar{\eta}). \qquad (5.13)$$

For the problem considered in this section the appropriate choice $M(\bar{\xi})$ and $N(\bar{\eta})$ is

$$M(\bar{\xi}) = 0, \qquad N(\bar{\eta}) = \lambda\delta^{-1}\exp(2b_1^1\bar{\eta}/\delta)H(-2\bar{\eta}). \qquad (5.14)$$

Hence, substituting (5.14) in (5.12) and using (5.11), we obtain

$$u = \left\{1 + \frac{x - x_0}{1 + x_0}\exp[-\tau(E_0/\rho_0)^{1/2}]\right\}H(\tau). \qquad (5.15)$$

Thus, when the material parameters E and ρ decrease with x according to (5.2) and (5.3) (with $m = -2$), the pulse propagates with constant speed. Also the amplitude of the displacement at $x = x_1 > x_0$ decreases from $(1 + x_1)/(1 + x_0)$ at $t = (\rho_0/E_0)^{1/2}(x_1 - x_0)$ to unity as $t \to \infty$.

Finally, it should be noted that the assumption that the rod is semi-infinite leads to the solutions having some features which are unsatisfactory from a physical viewpoint. However the solutions can certainly be regarded as giving a satisfactory description of the first stages in the propagation of waves along a finite rod.

REFERENCES

1. R. G. Payton, *Quart. J. Mech. and Appl. Math.* **19**, 83 (1966).
2. G. Eason, *Bull. Seism. Soc. Amer.* **57**, 1267 (1967).
3. H. Bremmer, *Comm. Pure Appl. Math.* **4**, 105 (1951).
4. R. Bellman and R. Kalaba, *J. Math. Mech.* **8**, 683 (1959).
5. L. M. Brekhovskikh, *Waves in Layered Media*. Academic Press, New York (1960).
6. C. Loewner, *NACA Tech. Note* 2065, **1** (1950).

Резюме — Рассматривают распространение волны в неоднородном эластичном стержне или плоской пластине. Определяющие уравнения записывают в виде таблицы и ищут преобразования, которые приведут систему к форме связанной с уравнением волны. Интегрирование системы тогда происходит немедленное. Выяснено, что такое преобразование возможно при условии, что функция, включающая параметры плотности и эластичности материала примет определенные многопараметрические формы. Эти параметры подходят по отношению к поведению разных неоднородных эластичных материалов. При помощи настоящего метода решается вопрос специфического исходного граничного значения.